Clause Set Cycles and Induction

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Background

We investigate several approaches to the automation of proof by induction. Currently we concentrate on the investigation of approaches based on cyclic clause sets in a classical setting. The automation of proof by induction is currently dominated by approaches originating from computer science. However there is a large discrepancy between the understanding of induction in computer science and mathematics.

Two Views of Induction

There are two orthogonal views of induction: the view of computer science and the view of mathemtatics.

Mathematics

- arithmetic theories
- natural numbers
- unprovability
- consistency

Computer Science

- automation
- datatypes
- algorithms
- efficiency

Automated Inductive Theorem Proving

Goal This field concentrates on the efficient automation of proving statements that involve *some form of induction*.

Relevance Automated inductive theorem proving is of great importance for the *formal verification of hardware and software*, and the formalization of mathematics.

Current State This field is characterized by a wealth of different approaches based on heuristics. These methods are often only empirically analyzed. There is *little overall progress*.

$\Longrightarrow$ Strengthening of formal foundations may be helpful

Research Program

We develop the formal understanding of approaches to automated inductive theorem proving by answering questions such as:

- What can a method prove and what can't it prove?
- Why does a certain method work well?
- How do methods related to each other?

We answer these questions by employing techniques and results from mathematical logic and in particular from proof theory.

Clause Set Cycles

Clause set cycles are a formal tool for the study of a family of approaches to automated inductive theorem proving originating in extensions of resolution calculi.

Definition: Let $S(n)$ be a clause set, then a clause set cycle for $S(n)$ is a clause set $R(n)$ such that

$$S(n) \models_{\mathrm{FOL}} R(n), \quad (1)$$

$$R(\mathsf{s}(n)) \models_{\mathrm{FOL}} R(n), \quad (2)$$

$$R(0) \models_{\mathrm{FOL}} . \quad (3)$$

A clause set $S(n)$ is refutable with clause set cycles if it admits a clause set cycle $R(n)$.

Σ_1-Bound

We first described the set of refutable clauses in terms of the quantifier complexity of the induction invariant that is required to refute a clause set.

Theorem 1: Let $S(n)$ be a clause set. If $S(n)$ is refutable with clause set cycles, then $S(n)$ is refutable with Σ_1-induction.

$\Longrightarrow$ Is Σ_1-induction necessary?

$\Longrightarrow$ Are clause set cycle refutations complete for Σ_1-induction?

Σ_1-Necessity

Theorem 2: The clause set $\mathsf{opt}_{\Sigma_1}(n)$ given below is refutable with clause set cycles, but it is not refutable with open induction.

$$x + 0 = x,$$
$$x + \mathsf{s}(y) = \mathsf{s}(x + y),$$
$$\mathsf{p}(0, \mathsf{s}(0)),$$
$$\mathsf{p}(x, y) \to \mathsf{p}(\mathsf{s}(x), y + y),$$
$$\neg \mathsf{p}(n, y).$$

Σ_1-Incompleteness

Conjecture 1: The clause set $\mathsf{inc}_{\Sigma_1}(n)$ given below is refutable with *open induction* but not with clause set cycles.

$$x + 0 = x,$$
$$x + \mathsf{s}(y) = \mathsf{s}(x + y),$$
$$n + (n + n) \neq (n + n) + n.$$

$\Longrightarrow$ Clause set cycles do not capture the required generalization.

Case Study: n-Clause Calculus

The n-clause calculus introduced by Kersani and Pelter in [KP13] enhances a superposition calculus by a cycle detection mechanism. An n-clause is a clause of the form

$$[\ \underbrace{t_1^1 \bowtie_1 t_1^2, \ldots, t_k^1 \bowtie_k t_k^2}_{\text{body}}\ \underbrace{\ |\ }_{\leftarrow}\ \underbrace{n \simeq r}_{\text{constraint}}\ ],$$

where $\bowtie_i \in \{\simeq, \not\simeq\}$, t_i^1, t_i^2 are individual terms for $i = 1, \ldots, k$, and r is a term representing a natural number. The cycle detection mechanism detects cyclic dependencies, *inductive cycles*, between n-clauses generated by the superposition calculus.

Theorem 3: Let $S(n)$ be a clause set. If $S(n)$ is refutable with inductive cycles, then $S(n)$ is refutable with clause set cycles.

Corollary: Every clause set refutable with inductive cycles, is refutable with Σ_1-induction.

By a straightforward transformation on proofs, Theorem 2 extends to the n-clause calculus. Furthermore, by the corollary above and by Conjecture 1, the n-clause calculus can be conjectured to be incomplete for Σ_1-induction.

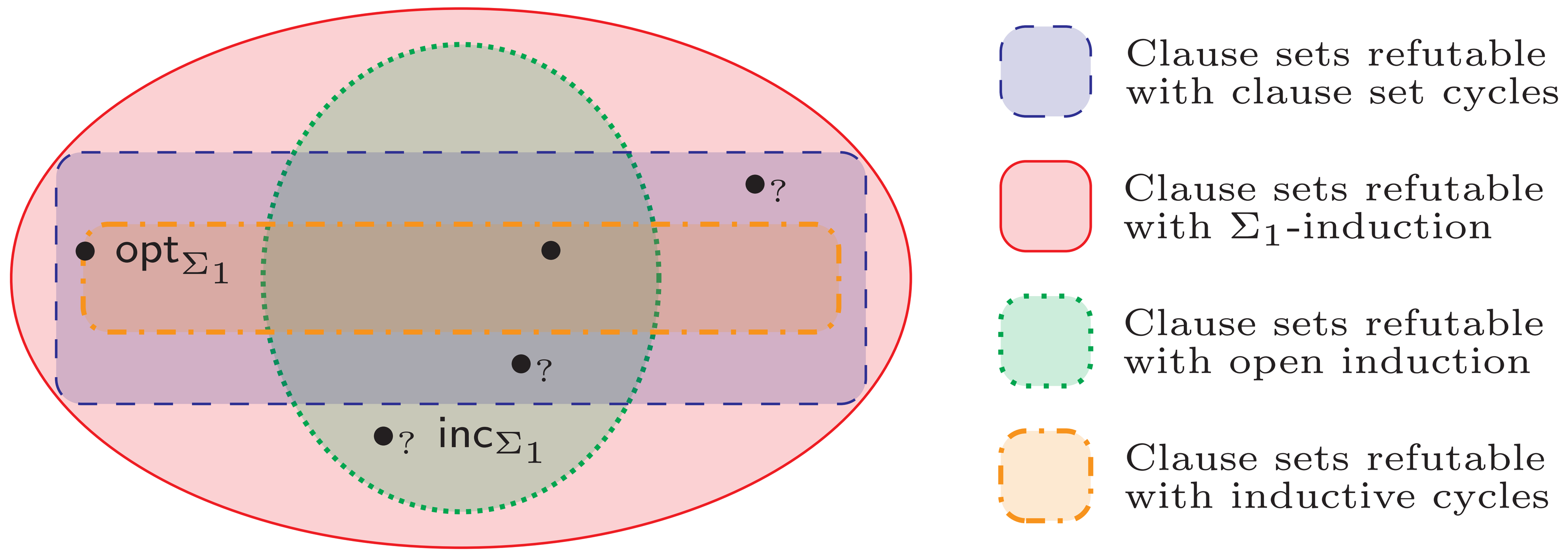

Figure 1: Summary of the relationships between the notions of induction.

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References

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